

Applications to quantum gravity from C*-bundles

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Abstract

Applications to quantum gravity of some results in C*-algebras are developed. We open by describing why algebra may be an integral aspect of quantum gravity. By interpreting the inner automorphisms of a C*-algebra as families of parallel transports on a C*-bundle, we define a notion of generalised connection for C*-bundles, implying that C*-bundle dynamical systems can be used to study a Dirac operator's spectrum and also the generator of the modular group. The fact that their dynamics preserves the bundle structure, means that C*-bundle dynamical systems are relativistic systems. We describe a 2-category to unify Einstein's two uses of the phrase general covariance, give concrete examples and construct a generally covariant quantum system. Motivations include the point of view of virtual points hiding in fibres of a C*-bundle over X and that physical geometry is intrinsically non-local.

1 Introduction: Generalised spaces

Algebra can be thought of as a laboratory, especially as gravity is a branch of geometry and geometry can be viewed through the window of algebra. The interpretations and applications of some previous results appear only in the form of passing comments in papers designated by operator algebra, so even those brief

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comments are unlikely to reach physicists working in quantum gravity who might be interested in them. In these notes we unify and develop those applications. We try to distinguish clearly between interpretational results and discussion, which is kept to a minimum. The algebraic technicalities are purposefully kept to a minimum.

Descartes, Grothendieck and Ehresmann each viewed geometry through the window of algebra. Ehresmann said that geometry is the study of differentiable categories (of which Lie groupoids are the most basic example but in the context of Quillen and Connes cyclic cohomology, a Morita-Rieffel category of imprimitivity bimodules can be viewed as a generalisation) and Grothendieck replaced the traditional continuous topologies with generalised topologies without points. Also from very early on, Heisenberg was asking for non-commuting configuration coordinates $[q_i, q_j] \neq 0$. Crane advocates that categories are an indispensable ingredient in quantum gravity. Of course, category theory is a topic in algebra so if quantum gravity is fundamentally categorical, then quantum gravity is fundamentally algebraic and so would benefit from more algebraic concepts. The distinguished direction of time often leads to theories breaking covariance but Connes and Rovelli made a physical interpretation of a distinguished state that already existed naturally in the intrinsic dynamical properties of a Von Neumann algebra [CR], and that indicated the emergence of a direction of time (pointing in the direction of increasing entropy). The path-integral formulation of quantum gravity involves various approaches to discretisation of space-time through choices of triangulations of the (traditionally smooth) space-time manifold and in [LC1] Crane explains that through algebraic methods the need to make that choice can be removed due to an a priori modelling of the quantum nature of the space-time. The spectral action principle [CC] by Connes and Chamseddine with important contributions from Schücker [S1, S2] involves a purely algebraic equivalence principle on a non-commutative Kaluza-Klein-like internal space. In this model the action is fully diffeomorphism invariant, albeit a classical action, this is another clear potential benefit of involving algebraic methods in quantum gravity.

Since quantum gravity involves trying to understand the nature of space-time, it will necessarily involve space-time on the smallest scales, or the Planck length scale. (Even though space-time-gravitational components of theories will not be tested directly on that level due to the negligible effect of the force, to understand gravity we first must understand more about what physical geometry really is.) It is very clear that Analytical calculus provided one of the greatest ever inventions but even so, Crane points out that it breaks down in the context of quantum space-time at the Planck length scale since it relies on infinitesimal distances as

part of the structure of this powerful mathematical tool. Fortunately there are already algebraic generalisations of calculus such as Connes' and Quillen's cyclic cohomology which draws heavily on Morita-Rieffel theory, which in itself has a rich geometrical significance.

Although it is traditional to pass to a non-commutative generalisation of a topological space by invoking the Gelfand-Naimark theorem and then viewing all non-commutative algebras as generalised algebras of functions, it is also true that $C_0(X)$ is the algebra of sections of a line bundle over X , so there is an argument to think of non-commutative algebras as generalised algebras of sections, where *the fuzzy points hide in the fibres* over (a tangible) X instead of in a fuzzy space " X ". The following candidate for a quantum space-time is not finished (although it was described in [M1]) but it has to be mentioned in order to explain the context of these notes.

Let E^0 be a C^* -bundle whose fibres are given by simple matrix algebras of varying dimension so that the enveloping algebra $C^*(E^0)$ is given by $A = \bigoplus M_{n_i}(\mathbb{C})$ $i = 1 \dots m$ or alternatively let A be an approximately finite algebra with direct limit A . The fibres A_i are all members of the same Morita equivalence class. Now consider the linking algebra B that A is embedded in, coming from the Morita equivalence bimodules (for instance, \mathbb{C}^2 is a Morita equivalence bimodule over $M_2(\mathbb{C})$ and \mathbb{C}). Just to illustrate for readers unfamiliar with this context, a typical element of a linking algebra,

$$e = \begin{pmatrix} e_{\gamma\gamma^*} & e_{\gamma} \\ e_{\gamma^*} & e_{\gamma^*\gamma} \end{pmatrix} \quad (1)$$

where $e_{\gamma\gamma^*} \in A_1$, $e_{\gamma^*\gamma} \in A_2$, $e_{\gamma} \in M_{AB}$ an A - B -bimodule and $e_{\gamma^*} \in M_{BA}$ a B - A -bimodule. [R].

Every space-time needs a tangent bundle, and this will come from the linear span of derivations from A into B : using basic category theory, one may define a representable sheaf from such a category of Morita equivalence bimodules to generalise the sheaf of sections of the tangent bundle. Further work is to check to what extent this satisfies the criteria of a Crane quantum space-time [LC1, LC2] and how closely it satisfies the axioms of an Algebraic quantum field theory and also how it relates to Rovelli's relativistic quantum mechanics.

In [L] following Barrett and Anandan, Lewandowski fully constructed a vector bundle together with a connection just from the data afforded by the holonomy group. Equivalently, a representation of a Lie groupoid on a vector bundle \mathcal{E} is equivalent to a choice of connection on \mathcal{E} . These statements imply that geometry can be fully described in a non-local way. Let M be a simply connected manifold with tangent bundle TM . When an integration of a tangent vector is performed,

these two things happen: an arrow in a differentiable category is obtained (i.e. the groupoid $M \times M$) and the algebra $C^*(TM)$ is deformed to a non-commutative algebra of observables $C^*(M \times M)$. This implies there is an intimate connection between category theory and non-commutative geometry, afterall, they are each topics from both geometry and algebra.

2 Generalised connections and geodesics

In short, a C^* -bundle is a bundle of C^* -algebras over a topological space where the enveloping algebra is a C^* -algebra $C^*(E^0)$ or A . For a full definition see [D] or [FD]. For Fell, Dixmier and many others, a C^* -bundle (E^0, π, X) is not equivalent to a fibre bundle with additional structures. In particular, it may be non-locally trivial. Also recall that the general linear groupoid $GL(\mathcal{E})$ of a vector bundle \mathcal{E} is the groupoid of all isomorphisms between pairs of fibres. Let $GL(E^0)$ denote the groupoid of isometric $*$ -isomorphisms between pairs of fibres of E^0 . Let A be represented on a separable Hilbert space \mathcal{H} .

Some simple examples of C^* -bundles:

- Examples 2.1.**
1. Let $A = \bigoplus_i M_{m_i}(\mathbb{C})$ so fibres of E^0 , denoted E_x^0 , $x \in X$, also denoted A_x , are simple matrix algebra summands $M_m(\mathbb{C})$ of varying dimension m .
 2. As the previous example except that all fibres are of equal dimension m .
 3. Let E^0 be a 1-dimensional C^* -bundle over a space X .

Recently, (reversible) C^* -bundle dynamical systems were defined:

Definition 2.2. [M3] A *reversible C^* -bundle dynamical system* $(E^0, A, \mathcal{G}_\sigma)$ is given by a C^* -bundle (E^0, A) and a 1-parameter covariance group \mathcal{G}_σ of C^* -bundle inner automorphisms.

where the elements f of \mathcal{G}_σ are of the form $f(a) = U_t a U_t^*$ with $U_t = e^{i\sigma t}$ for a possibly unbounded self-adjoint operator σ and with $a \in A$. Now let $a \in E_x^0$. Each f restricts to a family of fibrewise mappings $\alpha : E_x^0 \rightarrow E_y^0$, $\alpha_x(a) = u a u^*$ which from now on we call parallel transports.

In [M3] the interpretation was given to σ as an operator whose eigenvalues coincide with those of a Dirac operator D for a spectral triple (A, \mathcal{H}, D) with A and \mathcal{H} as above. Next we develop this as an application to quantum gravity with algebra.

In the special case that E^0 is a vector bundle (\mathcal{H} may be infinite dimensional if desired) then each U_t induces fibrewise isomorphisms or parallel transports

$\alpha : E_x^0 \rightarrow E_y^0$, $\alpha(a) = uau^*$ with $u = e^{i\sigma_x t}$ where from the set of components $\sigma_x \in A_x$ extended over all of X , one can infer the connection on the bundle. This leads to a notion of generalised connection:

Definition 2.3. Let $(E^0, \mathcal{G}_\sigma)$ be a reversible C^* -bundle dynamical system. Then a *generalised connection* is given by the operator σ .

In Connes' non-commutative counterpart of Einstein's equivalence principle, he constructs a curved space Dirac operator on a Riemannian spin manifold from an initial flat Dirac operator beginning from the data of a generalised diffeomorphism, that is, an inner automorphism of the spectral triple algebra. His fluctuations formula:

$$D^f = \sum_{\text{finite}} r_j L(\sigma_j) D L(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(A) \quad (2)$$

where L is the double valued lift of the automorphism group to the spinors. For the calculation of the spectral standard model action see [CC]. The result is the general form of the Dirac operator with arbitrary curvature and torsion:

$$D = \sum_i c_i \left(\frac{\partial}{\partial x_i} + \omega_i \right) \quad (3)$$

We infer that much if not all of the geometrical data needed for the connection on the spinor bundle can be constructed from the eigenvalues of D . Therefore, and especially in finite dimensions, the interpretation of σ as a certain kind of "generalised connection" (the kind defined precisely above) is arguably sensible.

In [M2] non-commutative geodesics were defined. For a C^* -bundle with isomorphic fibres, this notion of geodesic is given by the components $u = e^{i\sigma_x t}$ of each U_t . In [C] Connes first mentioned non-commutative geodesics as paths $\gamma = e^{i|D|t}$, where $|D|$ refers to eigenvalues of D . With the tangent sheaf replacing the "deformed tangent bundle" $M \times M$, then $u \in B$ or γ is exactly a path in the quantum space formalised by E^0 .

(A full treatment of the case of example 2.1(1) where fibres are not isomorphic will require generalising to irreversible C^* -bundle dynamical systems with 1-parameter semigroup of $*$ -endomorphisms implemented by partial isometries.)

In the context where the isotropy groups of the category $GL(E^0)$ are Lie groups, then σ_x will be an element of the Lie algebra of the group at x . (In order to study a space of generalised connections $\{\sigma\}$ a deeper study of the dynamical properties of Von Neumann algebras and modular groups might be needed,

especially following ideas of Bertozzini including in [BCL2]. Then it might be possible to calculate states for quantum gravity using algebraic methods.)

In [M3] it was shown that the following formula characterises a reversible C*-bundle dynamical system:

$$\Phi_{\hookrightarrow} = \sum_{m=1}^n \prod_{i=1}^m U_{g_i}, \quad g_i \in \mathcal{G}_\sigma, \quad i = 1..m, \quad m = 1..n. \quad (4)$$

Φ_{\hookrightarrow} resembles a partition function since it is a sum over geometrical states and the product consists of parallel transports. (More work is needed to make this precise but the fundamental excitations $u_{g^*g \in X \times X} \in A_x$ should correspond to the components σ_x .)

3 General covariance 2-categories

A C*-bundle dynamical system satisfies an additional condition called the covariance condition. This is the property that the inner automorphisms f respect the bundle structure. For dynamical systems it means that a physical configuration space is singled out (and other mathematical consequences that we treated in [M3]). Let G denote the pair groupoid $X \times X$ and let \mathcal{G} denote the group of bisections of G , which is identified with the group of homeomorphisms $f_0 \in \text{Homeo}(X)$. The physical significance of the covariance condition is that in given physical theory, if X is a space-time and if $\mathcal{G} = \text{Homeo}(X)$ is replaced by the symmetry group of the theory (perhaps adding smoothness conditions) then the theory is covariant with respect to the group \mathcal{G} . So the inner automorphisms of a C*-bundle covariant with respect to all coordinate transformations or diffeomorphisms will give rise to a generally covariant theory whereas if the symmetry group \mathcal{G} of the theory is the Poincaré group then the theory will be Poincaré covariant. Since C*-dynamical systems are quantum mechanical systems, this means that a C*-bundle dynamical system is a relativistic quantum system.

Definition 3.1. C*-bundle inner automorphisms must preserve the bundle structure. The corresponding condition,

$$\pi \circ f = f_0 \circ \pi \quad (5)$$

is called the *covariance condition*.

By *general covariance*, Einstein meant two things: (i) the theory is diffeomorphism invariant, the physics being independent of the choice of coordinates and he also meant that (ii) there was no given initial background geometrical data,

the curvature of space-time originated from fluctuations away from the initial flat metric. A 2-category can capture general covariance where the first meaning (i) comes into the 1-morphisms and the second meaning (ii) is captured by the 2-morphisms:

By *general covariance 2-category* we mean a 2-category \mathcal{C} with an object space $\text{Ob}(\mathcal{C})$ with a disjoint union operation to model a space-time, with 1-morphisms to model parallel transports in a covariant way and 2-morphisms to model fluctuations of the gravitational field implementing the equivalence principle.

(Ultimately, a general covariance 2-category should include an inverse category of partial maps so that bundles with non-isomorphic fibres can be included).

Example 3.2. Let E^0 be a C^* -bundle with isomorphic fibres and let $GL(E^0)$ be the groupoid of all isometric $*$ -isomorphisms between pairs of fibres (E_x^0, E_y^0) . Let $A = C^*(E^0)$ be represented on a separable Hilbert space. The following describes a general covariance 2-category \mathcal{C} .

- Objects are the fibre C^* -algebras A_i of E^0 . Ob has a direct sum operation.
- 1-arrows are all the elements of $GL(E^0)$, $a \mapsto uau^*$ with $u = e^{is_x t}$ with s_x the component at x of a possibly unbounded self-adjoint operator s , for each choice of u .
- 2-arrows are given by assignments $s_x \mapsto us_x u^*$

Example 3.3 (General covariance 2-group). The previous example gives rise to a 2-group such that 1-arrows are the global bisections of $GL(E^0)$, which, exactly because they are bisections, satisfy $\pi \circ f = f_0 \circ \pi$, (the mathematical reason for this is made clear in [M3].) 2-arrows are fluctuations: $s \mapsto UsU^*$ to be compared with formula 2. Note that a 2-arrow is almost iterative of a 1-arrow.

Example 3.4. 1. Objects: the same as in the previous example.

2. 1-arrows: Morita equivalence bimodules.
3. 2-arrows: bimodule isomorphisms.

Example 3.4 is a coarse-grained version of example 3.2 since:

each 1-arrow of 3.4 is given by the linear span of the commutators $[s, c]$ over $c \in A_x \oplus A_y$ for any s_x and is the A_x - A_y -bimodule Ω_x^1 which we also denote E_{xy} .

On the 2-arrow level, $\text{span}_{c \in A_x \oplus A_y} [us_x u^*, c] = uE_{xy}u^*$ because for all $c' \in A_x \oplus A_y$ there exists a c such that $[us_x u^*, c] = u[s, c']u^*$, and $e \mapsto ueu^*$ for all $e \in E_{xy}$ defines a bimodule isomorphism.

Proposition 3.5. *A (reversible) C^* -bundle dynamical system together with a general covariance 2-group is a generally covariant quantum mechanical system.*

Example 3.6. Any spectral C^* -category (see [M2] for the definition) with isomorphic objects and any spectral triple of an equivalent description.

It remains to verify if these systems satisfy requirements asked for in the literature on and related to covariant quantum mechanics.

4 Inner categorification

The only news in this section is just discussion.

The procedures in “inner categorification” of spectral triples [M2] (resulting in spectral C^* -categories) came from the point of view on non-local geometry explained in the last paragraph of the first section together with the view that fuzzy points might be hiding in the fibres rather than in the base manifold, reflecting the bundle-like nature of a Riemannian spin manifold. Connes explains that a curved Dirac operator only arises from formula 2 in the non-commutative case, so it was suggested in [M2] that space-time curvature might arise from non-commutativity of space-time. More studies involving σ_x as a Lie algebra element might shed light. In the case of a C^* -bundle dynamical system with a line bundle E^0 , the fibre is \mathbb{C} , the bundle endomorphisms form a group and so the system is reversible. In contrast, since not all fibres are given by \mathbb{C} , a dynamical system with E^0 from example 2.1(1) will involve endomorphisms between fibres that are not invertible, which means that some of the dynamics will be irreversible and the question arises if the time irreversibility we observe has emerged from the above curving effect of the non-commutativity in the space E^0 (especially if the dynamical system is based on a Von Neumann crossed product [CR]).

With inner categorification, together with the details of spectral triple morphisms appearing in [BCL4], [BCL1], [BCL3] and other works by those authors and others, together with a bundle dynamical perspective, it should be possible to construct an algebraic counterpart to TQFTs in a canonical way. In particular, these C^* -categories are concrete, which means they already possess a representation or functor into a category of Hilbert spaces and bounded linear maps, Hilb .

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